Junior prolem

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September, $8th$, 2013

J276. Find all positive integers m and n such that

$$10^n - 6^m = 4n^2$$

Solution 1. First, we prove a lemma.
Lemma 1. If $a \in \mathbb{N}, a \geq 2$ then $10^a > 6^a + 4a^2$.
Proof. If $a = 2$ then $10^2 > 6^2 + 4 \cdot 2^2$. Assume that the inquality still true with $n = k$ ($k \in \mathbb{N}, k \geq 2$), that means $10^k > 6^k + 4k^2$.
We will prove that $10^{k+1} > 6^{k+1} + 4(k+1)^2$. The inequality is equivalent to $10(10^k - 6^k) + 4 \cdot 6^k > 4k^2 + 8k + 4$.
We have $10(10^k - 6^k) + 4 \cdot 6^k > 10 \cdot 4k^2 + 4 \cdot 6^k$. Since $k \geq 2$ then $9 \cdot k^2 > 8k + 4$. Thus, $10 \cdot 4k^2 + 4 \cdot 6^k > 4k^2 + 8k + 4$.
Thus, $10^a > 6^a + 4a^2$.

If $n = 1$ then $m = 1$. If $n = 2$ or $n = 3$ then there is no such m.
If $n \geq 4$, from the lemma 1 we obtain that $6^m > 6^n$. Hence $m > n \geq 4$.

Case 1. If n is odd then $4n^2 \equiv 4 \pmod{16}$ and $10^n \equiv 0 \pmod{16}$ since $n \geq 4$. Therefore $6^m \equiv 12 \pmod{16}$. It follows that $m = 2$, a contradiction.
Case 2. If n is even, we let $n = 2n_1$ ($n_1 \in \mathbb{N}, n_1 \geq 2$). We also have $4n^2 \equiv 0, 1, 2, 4 \pmod{7}$.
We will prove that m is even.

1. If $3|n_1$ then $3|4n^2$ and $3|6^m$ so $3|10^n$, a contradiction.

2. If $n_1 = 3k + 1$ ($k \in \mathbb{N}, k \geq 1$) then by Fermat's Little Theorem we have $10^n = 10^{6k+2} \equiv 10^2 \equiv 2 \pmod{7}$. If m is odd then $6^m \equiv 6 \pmod{7}$, a contradiction since $10^n - 4n^2 \equiv 0, 1, 2, 5 \pmod{7}$. Thus, m is even in this case.

3. If $n_1 = 3k + 2$ then $10^n = 10^{6k+4} \equiv 4 \pmod{7}$. If m is odd then $6^m \equiv 6 \pmod{7}$, a contradiction since $10^n - 4n^2 \equiv 0, 2, 3, 4 \pmod{7}$.

Thus, m is even. Let $m = 2m_1$ ($m_1 \in \mathbb{N}, m_1 > 2$). The equation is equivalent to

$$(10^{n_1} - 6^{m_1})(10^{n_1} + 6^{n_1}) = 16n_1^2 \qquad (1)$$

Since $m > n$ then from (1) we obtain

$$2^{2n_1}(5^{n-1} - 2^{m_1 - n_1} \cdot 3^{m_1})(5^{n_1} + 2^{m_1 - n_1} \cdot 3^{m_1}) = 16n_1^2$$

Let $n_1 = 2^q \cdot k$ $(q, k \in \mathbb{N},\ 2 \nmid k)$. then the equation is equivalent to

$$(5^{n-1} - 2^{m_1-n_1} \cdot 3^{m_1})(5^{n_1} + 2^{m_1-n_1} \cdot 3^{m_1}) = \frac{2^{2q+4}}{2^{2^{q+1} \cdot k}} \cdot k^2$$

Since $m > n$ then $m_1 > n_1$. Therefore $LHS \equiv 1 \pmod 2$. Thus, $2^{2q+4} = 2^{2^{q+1} \cdot k}$ or $q + 2 = 2^q \cdot k$.
It is easy to prove by induction that if $q \geq 1, k \geq 3$ then $2^q \cdot k > 2q + 4$. It follows that $k = 1$, that means $LHS = 1$, a contradiction.
Thus, the only solution is $\boxed{(m, n) = (1, 1)}$.

Solution 2. From the solution 1, we have that if $n \geq 4$ then $m > n \geq 4$ and n is even. Let $n = 2^{q+1} \cdot k,\ (q, k \in \mathbb{N},\ 2 \nmid k)$. The equation is equivalent to

$$2^{2q+4}(2^{n-2(q+2)} \cdot 5^n - k)(2^{n-2(q+2)} \cdot 5^n + k) = 2^m \cdot 3^m$$

From here we obtain $m = 2q + 4$ but $m > n$ or $q + 2 > 2^q \cdot k$. Thus, $k = 1, q = 0$ or $k = 1, q = 1$ which gives $n = 2$ or $n = 4$,a contradiction.
Thus, $\boxed{(m, n) = (1, 1)}$.